International School on Aperiodic Crystals

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Mathematical refresher

Gervais Chapuis, LCr
EPFL

The Dirac delta function δ(x)

$$\delta(x-a) = \begin{cases} 0 & for \quad x \neq a \\ \infty & for \quad x = a \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-a)dx = 1$$

$$\delta(x) = \int_{-\infty}^{\infty} \exp(2\pi \mathrm{i} xy)\mathrm{d}y$$

The convolution product

$$f(x) * g(x) \equiv \int_{-\infty}^{\infty} f(X)g(x-X)dX$$

$$f(x) * g(x) = g(x) * f(x)$$

$$f(x) * \delta(x) = f(x) \qquad f(x) * \delta(x-a) = f(a)$$

In vector form

$$f(\mathbf{x}) * g(\mathbf{x}) = \int_{-\infty}^{\infty} f(\mathbf{X})g(\mathbf{x}-\mathbf{X})d\mathbf{X}$$

The Fourier transform

The Fourier transform (FT) of a function $f(x)$ is indicated by $\mathcal{F}\,[\,f(x)\,]$

$$\mathcal{F}\,[f(x)] \equiv F(u) = \int_{-\infty}^{\infty} f(x) \exp(2\pi \mathrm{i} u x)\, \mathrm{d}x$$

The inverse FT ($\mathcal{F}^{-1}$) of a Fourier transformed function gives the original function

$$f(x) = \mathcal{F}^{-1}\{\mathcal{F}\,[f(x)]\} = \mathcal{F}^{-1}\{F(u)\}$$
$$= \int_{-\infty}^{\infty} F(u) \exp(-2\pi \mathrm{i} u x)\, \mathrm{d}u$$

The Fourier transform (2)

The FT can be generalised in higher dimensional space

$$F(\mathbf{u}) = \int_{-\infty}^{\infty} f(\mathbf{x}) \exp(2\pi \mathrm{i} \mathbf{u} \cdot \mathbf{x}) \, \mathrm{d}\mathbf{x}$$

$$F(u,v,w) = \iiint_{-\infty}^{\infty} f(x,y,z) \exp\{2\pi \mathrm{i}(ux + vy + wz)\} \mathrm{d}x \mathrm{d}y \mathrm{d}z$$

$$f(x,y,z) = \iiint_{-\infty}^{\infty} F(u,v,w) \exp\{-2\pi \mathrm{i}(ux + vy + wz)\} \mathrm{d}u \mathrm{d}v \mathrm{d}w$$

Some simple rules concerning FT

$$\mathcal{F}\left[f(x)\right] = F(u)$$

$$\mathcal{F}\left[f(-x)\right] = F(-u)$$

$$\mathcal{F}\left[\delta(x)\right] = 1$$

$$\mathcal{F}\left[f(x-a)\right] = \exp(2\pi \mathrm{i} a u) F(u)$$

$$\mathcal{F}\left[\delta(x-a)\right] = \exp(2\pi \mathrm{i} u a)$$

$$\mathcal{F}\left[f^*(x)\right] = F^*(-u)$$

$$\mathcal{F}\left[f(x)+g(x)\right] = F(u)+G(u)$$

$$\mathcal{F}\left[f(x)\cdot g(x)\right] = F(u)*G(u)$$

$$\mathcal{F}\left[f(x)*g(x)\right] = F(u)\cdot G(u)$$